Concept: Keep your eye on the ball. Sometimes we don't need to find the values of all the variables in a problem to solve the problem.

Fortunately, we can find this product without ever finding the values of b that satisfy $b^2 - 2b - 4 = 0$. Suppose b_1 and b_2 are the solutions to this equation, so we have $y_1 = 64^{b_1}$ and $y_2 = 64^{b_2}$. Then, we have

$$y_1y_2 = 64^{b_1} \cdot 64^{b_2} = 64^{b_1+b_2}.$$

Since b_1 and b_2 are the solutions to the equation $b^2 - 2b - 4 = 0$, Vieta tells us that $b_1 + b_2 = -(-2)/1 = 2$. So, we have $y_1y_2 = 64^2$.

Similarly, from $a = \log_{225} x$, we have $x = 225^a$. Since $a = 4 - b$, we have $x = 225^{4-b}$. So, from our two values of b, namely b_1 and b_2, we get two values of x, which we call x_1 and x_2, and we have

$$x_1x_2 = 225^{4-b_1} \cdot 225^{4-b_2} = 225^{8-(b_1+b_2)} = 225^6.$$

So, we have $\log_{30} x_1y_1x_2y_2 = \log_{30}(64^2)(225^6) = \log_{30}(2^{12} \cdot 3^{12} \cdot 5^{12}) = \log_{30} 30^{12} = 12.$ □

Exercises

13.4.1 If r and s are the roots of $3x^2 - 16x + 12 = 0$, then find $\log_2 r + \log_2 s$.

13.4.2 Given that $\log_{10} 17 = r$ and $\log_{10} 2 = s$, find the value of $\dfrac{\log_{10} \frac{1600}{17}}{\log_{10} 136}$ in terms of r and s.

13.4.3 Find all t such that $2\log_3(1 - 5t) = \log_3(2t + 5) + 2$.

13.4.4 Let $x_1, x_2, \ldots, x_n$ be a geometric sequence of positive real numbers. Prove that for any positive number b, with $b \neq 1$, the sequence $\log_b x_1, \log_b x_2, \ldots, \log_b x_n$ is an arithmetic sequence.

13.4.5 Suppose the ordered pair (x, y) satisfies $\dfrac{\log_{10}(xy)}{\log_{10}\left(\frac{x}{y}\right)} = \dfrac{1}{2}$. If y is increased by 50%, by what fraction must x be multiplied to keep this equation true? *(Source: ARML)*

13.4.6 A line $x = k$ intersects the graph of $y = \log_5 x$ and the graph of $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Find k. *(Source: AHSME)*

13.4.7 For all integers n greater than 1, define

$$a_n = \frac{1}{\log_n 2002}.$$

Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. What is $b - c$? *(Source: AMC 12)* **Hints:** 219

13.4.8 Compute the integer k, where $k > 2$, for which

$$\log_{10}(k-2)! + \log_{10}(k-1)! + 2 = 2\log_{10} k!.$$

(Source: ARML) **Hints:** 86